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A modified sequential approach for solving inverse heat conduction problems

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Abstract

A modified sequential approach is proposed to improve the performance of the sequential function specification method for inverse heat conduction problems (IHCPs). There are two essential procedures in this study: the first, derives the sequential algorithm and then performs the preliminary estimation; the second, also a key finding in this study, proposes the modified algorithm to eliminate the leading error caused by adding the use of future information in the process of preliminary estimation. One example in this study for solving the unknown source, the proposed method effectively reduces the average relative error from 17.82% to 2.65% when a 10% random measurement error is considered.

Keywords: Inverse; Sequential; Function specification method; IHCP; Leading error; Heat source

1. Introduction

For the requirement of industrial designs and manufacturing procedures, some essential thermodynamic parameters that are not obtainable from direct measurements. These parameters, for instance, the strength of heat source, the interior or boundary temperature, the surface heat flux, the thermal property, and the geometry boundary, can be obtained by numerical computation when some measurable data are implemented. The determination of the unknown thermodynamic parameters is classified into the field of the inverse heat conduction problem (IHCP).

The IHCP is an ill-posed problem, namely a small input disturbance would lead to a large output error or improper dispersion. Therefore, diminishing the instability caused by the ill-posed feature is crucial for estimating the undetermined parameters in an inverse problem. Over the past few decades, some authors had been presented several methods in improving the stability of IHCP [1–4]. Where the method of sequential function specification associated with several future temperatures, proposed by Beck et al. [1], indicates that the use of several future times, $r > 1$, greatly improves the stability of ill-posed problem for IHCP and substantially reduces the sensitivity to measurement errors. The formulation has been used widely in many studies of inverse problem. For instance, Yang [5–7] employed the method to determine interior source and mix-typed boundary conditions, Videcoq and Petit [8] utilized it to propose a model reduction approach, instead of using a detailed model of large size, to estimate the inverse heat flux problem, Kim and Lee [9] applied it to estimate the time-varying heat transfer coefficient for the nonlinear IHCP, and Chantasiriwan [10] applied it to estimate the time-dependent Biot number. Additionally, some other authors also applied the method to determine the unknown parameters of inverse problem and obtained many referable results [11–13]. The advantages of the sequential function specification method (SFSM) are that it is easy to analyze, it does not require any iterative processes or prior information about the unknown function, and it can determine all unknown parameters

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Nomenclature

$[A]$	coefficient matrix of temperature
$[B]$	transform matrix
$[D]$	matrix, $[D]=[A]^{-1}$
$\{T\}$	temperature vector
M	sensitivity matrix
Q	heat source
T	temperature
T_0	initial temperature
Y	measured temperature
r	number of future time steps
t	temporal coordinate
u	denote the component of $[D]^{k+1}\{T^{n-1}\}$
x	Cartesian coordinate
Greek symbols	
α	approximate slope
β	estimated amplitude of step function
φ	heat source
γ	denote the component of $\sum_{a=1}^{k+1}[D]^a[B]$
λ	random real number between 1 and −1
σ	standard deviation of measurement error
$\hat{\psi}$	estimated source strength
Δt	increment of temporal domain
Δx	increment of spatial domain
Θ	vector
Subscripts	
i	spatial coordinate
q	spatial coordinate of heat source
Superscripts	
a	index
k, n	temporal coordinate
r	number of future time steps

simultaneously and, as a result, it provides good computational efficiencies. The basic procedures of the SFSM are to use the future temperature information and the temporal assumption that the undetermined parameter is, for example, a constant, a time-dependent linear variation, or a parabolic form, over the future time steps. Such an operation is an efficacious one for improving the stability, but comparatively, some factors still need to be discussed as following:

- The temporary assumption of the undetermined parameter is a constant, a time-dependent linear variation, or some other forms over the future time steps, which might not match the character of the undetermined parameter exactly, would cause a systematic "leading error" that caused due to the use of future temperatures to compute the present parameter. Consequently, the leading error is different from the error caused by random measurement error.
- The leading error exists when the future information is used. Even if the assumed condition is exactly equal to the undetermined parameter over the used interval of future times, the estimated error will still appear on those time steps before the slope of undetermined parameter is variant.
- The stability of solution of the IHCP is improved progressively by increasing the number of future times. In contrast, the leading error also rises depending closely on the increase of number of future times. Consequently, for avoiding the large leading error appearance, Beck et al. [1] refer to that the value of future time number is commonly chosen to be about 3 or 4. Nevertheless, when the measurement error is considered, more future times is often essential to obtain a stable estimation.

Based on the above features, the modified algorithm for sequential function specification method (MSFSM) is proposed in this paper to provide an expectantly accurate and stable estimation of the solution. There are two distinct aspects of this study: One is to derive the sequential algorithm in order to launch the calculation of undetermined parameter. The other is to propose the modified algorithm to eliminate the leading error that is caused by incorporating future times in the SFSM. An implication from the different viewpoint is that by using the proposed method can avoid having to use only a small number of future time steps, and hence induces a significant instability of the solution, solely for the purpose of avoiding the leading error. Consequently, the accuracy and stability can be improved effectively by using the proposed method. Four examples are applied in this paper to demonstrate the characteristics and performances of the proposed method.

The approach in this study is first to utilize finite difference method (FDM) to discrete the governing equation firstly, and then to derive an inverse sequential algorithm. Next, the formulation of future time and the temporary assumption of constant source over an arbitrary number of future time steps are applied in order to get a stable, but inaccurate, result. Finally, a modified procedure is employed to eliminate the leading error and obtain an accurate and stable estimation. It should be noted that the assumption of a constant source is not necessary to be applied: other assumptions, such as a time-dependent linear or parabolic function, are possible and have a similar effect.

The main features and characteristics of this study are summarized as follows:

- The sequential function specification method is a robust method for solving undetermined parameters in the IHCP. A problem, however, is that a high stability of estimation can cause a large leading error.
- For improving the stability of estimation when the measurement error is considered, it is indispensable to use more future information. An additional effect, however, is that the leading error enlarges gradually upon increasing the number of future time steps.
- The method proposed in this study can eliminate the leading error and lead to an accurate estimation. Four examples are employed to demonstrate the utility of this method.
- Owing to superior stability can be obtained by increasing the number of future times and since the leading error can be eliminated effectively by this proposed method, so that the accurate and stable estimation can be obtained even if the measurement error is not slight.
- The characteristics of this proposed method are that no prior ideal conditions and iterative processes are need, good computational efficiency and reliability ensues, and that it can be applied to estimating an undetermined parameter having an arbitrary functional form.

2. Mathematical formulation of the proposed method

Consider a one-dimensional object having the dimensionless length L. Both of the two ends of the object are adiabatic. The initial temperature of the object and the surrounding temperature are T_0. As a specific time $t = 0$, one heat source Q that is a function of time acts at the coordinate position $x = x_q$. Based on the description, the dimensionless governing equation and the initial and boundary conditions can be presented as follows:

$$\frac{\partial^2 T}{\partial x^2} + Q(t)\delta(x - x_q) = \frac{\partial T}{\partial t}, \tag{1}$$

$$T(x,0) = T_0, \quad 0 \leqslant x \leqslant 1, \tag{2}$$

$$\frac{\partial T}{\partial x} = 0, \quad x = 0, 1. \tag{3}$$

The subscript q denotes the spatial coordinate of the heat source. For the sake of simplification, all the thermal properties in this problem are considered to be constant. In order to perform the numerical operation, the proposed method utilizes the FDM to discretize both spatial and temporal coordinate in this study.

$$\frac{T_{i-1}^k - 2T_i^k + T_{i+1}^k}{(\Delta x)^2} + Q(t)\delta(x - x_q) = \frac{T_i^k - T_i^{k-1}}{\Delta t}. \tag{4}$$

In Eq. (4), the subscript i denotes the grid position in x coordinate and superscript k denotes the number of the time index. By rearranging Eq. (4), a vector expression, Eq. (5), is readily derived in which $[A]$ and $[B]$ are the coefficient matrices of temperature distribution vector $\{T\}$ and the undetermined source vector $\{\varphi^k\}$.

$$\begin{aligned}\{T^k\} &= [A]^{-1}\{T^{k-1}\} + [A]^{-1}[B]\{\varphi^k\} \\ &= [D]\{T^{k-1}\} + [D][B]\{\varphi^k\},\end{aligned} \tag{5}$$

where

$$[D] = [A]^{-1}.$$

A numerical operation approximation [6] needs to be established to solve the unknown parameter. For this reason, a matrix form of the numerical operation at a specific time gird n is required and is shown as Eq. (6). The item r is the number of used future times and $\{T^n\}, \{T^{n+1}\}, \ldots, \{T^{n+r-2}\}$, $\{T^{n+r-1}\}$ indicate the temperature distributions of time t^n, $t^{n+1}, \ldots, t^{n+r-2}$, t^{n+r-1}, respectively. It is very significant that the construction of Eq. (6) is really improper and not the anticipated form for the estimation, as a result of the large multitude of elements, and that it will lead to an inefficient computation. In other word, Eq. (6) must be improved in order to obtained a practicable operation:

$$\begin{Bmatrix} \{T^n\} \\ \{T^{n+1}\} \\ \vdots \\ \vdots \\ \{T^{n+r-2}\} \\ \{T^{n+r-1}\} \end{Bmatrix} = \begin{bmatrix} [D] & 0 & \cdots & \cdots & \cdots & 0 \\ [D]^2 & [D] & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ [D]^{r-1} & [D]^{r-2} & \cdots & & \ddots & [D] & 0 \\ [D]^r & [D]^{r-1} & \cdots & \cdots & \cdots & [D]^2 & [D] \end{bmatrix} [B] \begin{Bmatrix} \varphi^n \\ \varphi^{n+1} \\ \vdots \\ \vdots \\ \varphi^{n+r-2} \\ \varphi^{n+r-1} \end{Bmatrix} + \begin{bmatrix} [D] \\ [D]^2 \\ \vdots \\ \vdots \\ [D]^{r-1} \\ [D]^r \end{bmatrix} \{T^{n-1}\}. \tag{6}$$

The temperature data $T_i^n, T_i^{n+1}, \ldots, T_i^{n+r-2}, T_i^{n+r-1}$, measured at a position $x = x_i$ and various time girds, are used to compute the unknown source. Consequently, the following component equations can be listed:

$$T_i^n = u_i^0 + \gamma_i^n \varphi^n, \tag{7}$$

$$T_i^{n+1} = u_i^1 + \gamma_i^{n+1}\varphi^n + \gamma_i^n \varphi^{n+1}, \tag{8}$$

...

$$T_i^{n+r-2} = u_i^{r-2} + \gamma_i^{n+r-2}\varphi^n + \gamma_i^{n+r-3}\varphi^{n+1} + \cdots + \gamma_i^{n+1}\varphi^{n+r-3} + \gamma_i^n \varphi^{n+r-2}, \tag{9}$$

$$T_i^{n+r-1} = u_i^{r-1} + \gamma_i^{n+r-1}\varphi^n + \gamma_i^{n+r-2}\varphi^{n+1} + \cdots + \gamma_i^{n+1}\varphi^{n+r-2} + \gamma_i^n \varphi^{n+r-1}, \tag{10}$$

where $u_i^k = e_i \cdot [D]^{k+1}\{T^{n-1}\}$, $\gamma_i^{n+k} = e_i \cdot \sum_{a=1}^{k+1}[D]^a[B]$ and k is an integer between 0 and $r-1$, and e_i is the unit row vector with a unit at ith component.

To stabilize the computational results, the temporary assumption of a constant source is assumed over the used future times. When $t = t^n$, the unknown parameters φ^n, $\varphi^{n+1}, \ldots, \varphi^{n+r-2}$, φ^{n+r-1} in Eqs. (7)–(10) are equal and defined as Eq. (11):

$$\varphi^n = \varphi^{n+1} = \cdots = \varphi^{n+r-2} = \varphi^{n+r-1}. \tag{11}$$

Applying the assumption in (11), Eqs. (7)–(10) can be rearranged to derive Eqs. (12)–(15). It is worthy to be emphasized that even though Eq. (11) is a temporarily inexact assumption, it is effective to stabilize the computational result. Comparatively, such assumption would lead to a specific error that grows closely in relation to the amount of future information used.

$$T_i^n = u_i^0 + \gamma_i^n \varphi^n, \tag{12}$$

$$T_i^{n+1} = u_i^1 + (\gamma_i^{n+1} + \gamma_i^n)\varphi^n, \tag{13}$$

...

$$T_i^{n+r-2} = u_i^{r-1} + (\gamma_i^{n+r-2} + \gamma_i^{n+r-3} + \cdots + \gamma_i^{n+1} + \gamma_i^n)\varphi^n, \tag{14}$$

$$T_i^{n+r-1} = u_i^{r-1} + (\gamma_i^{n+r-1} + \gamma_i^{n+r-2} + \cdots + \gamma_i^{n+1} + \gamma_i^n)\varphi^n. \tag{15}$$

For numerical computation, Eqs. (12)–(15) can be rearranged into the following form:

$$\Theta = M\psi, \tag{16}$$

where

$$\Theta = \left\{ \begin{array}{c} T_i^n - u_i^0 \\ T_i^{n+1} - u_i^1 \\ \vdots \\ T_i^{n+r-2} - u_i^{r-2} \\ T_i^{n+r-1} - u_i^{r-1} \end{array} \right\},$$

$$M = \left\{ \begin{array}{c} \gamma_i^n \\ \gamma_i^{n+1} + \gamma_i^n \\ \vdots \\ \sum_{k=2}^{r} \gamma_i^{n+k-2} \\ \sum_{k=1}^{r} \gamma_i^{n+k-1} \end{array} \right\}, \quad \psi = \{\varphi^n\}.$$

Then the undetermined strength of heat source can be estimated in each time step by applying the linear least-squares error method

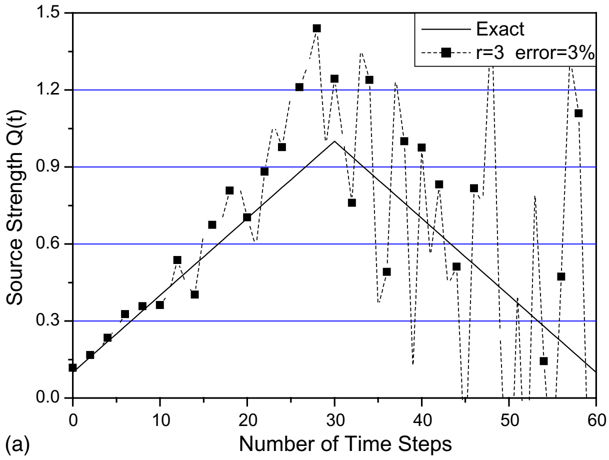
$$\hat{\psi} = (M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}\Theta, \tag{17}$$

where $(M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}$ is defined as a reverse matrix. Consequently, the iterative procedure in the problem can be avoided. Moreover, the rank of reverse matrix is available to indicate that the least-squares solution can be approximated.

3. Modified sequential algorithms based on preliminary estimation

The results of many authors' studies show that the use of future information could mitigate the fluctuant estimation resulted from an ill-posed feature when the input error is considered. Additionally, the greater the number of future time steps causes the estimation in the whole time domain to be more smooth and stable. As the prior description, combining Eqs. (12)–(15) with the future temperatures and the assumption of the constant source strength results in a stable computational result that is obtained by the linear least-squares error method. Nevertheless, because the assumption of the constant source strength is different from the exact form of the undetermined source, the variance grows increasingly with the number of future time steps. In other word, the leading error increases as the number of future time step is added. On the other hand, contradictorily, using fewer future times to prevent producing significant leading error might lead to a fluctuant estimation, as displayed in Fig. 1(a).

Fig. 1(b) shows the results of estimation with various numbers of future time steps. It is very clear that the number of future time has a positive effect on the estimating stability and a negative effect on the leading error. This phenomenon would not appear when the temporary assumption about the function of undetermined parameter in the computational procedure is exactly matching. As a matter of fact, this anticipation is impracticable. Consequently, the simple assumption of a constant parameter is commonly used, as it is in this study. The following procedures are to propose the formulations for modifications having various functional forms of heat source.



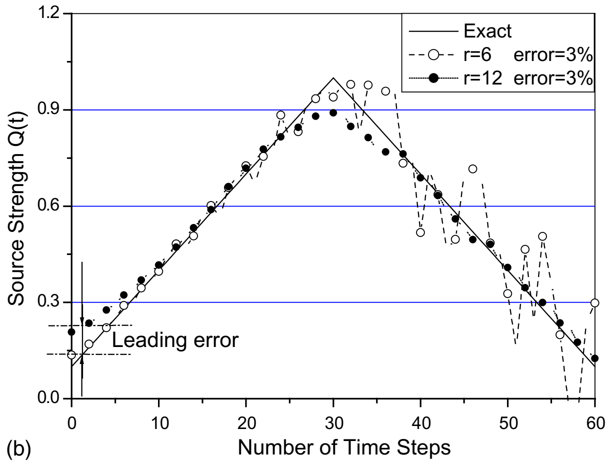


Fig. 1. The estimation in sequential function specification method (SFSM) with 3% measurement error (a) $r = 3$; insufficient future time steps (b) $r = 6$, 12.

3.1. Case A: triangular source

Fig. 2(a) shows the computational result of an undetermined triangular source by SFSM with 3% random measurement error and $r = 15$. This result provides a description about the existence of both considerable leading error and slight fluctuation of the solution. As mentioned above, the contradictory finding is that a more stable solution would lead to a more significant leading error. In this situation, the proposed method is first to find the piecewise approximate slope. Consequently, as Fig. 2(a) shown, two fitting lines $L1$ and $L2$ are constructed based on the result of the preliminary estimation and the correlation of the estimated undetermined source over the piecewise interval can be listed as follows:

$$\varphi^{n+1} = \varphi^n + \Delta t \times \alpha, \tag{18}$$

$$\varphi^{n+2} = \varphi^n + 2\Delta t \times \alpha, \tag{19}$$

$$\cdots$$

$$\varphi^{n+r-2} = \varphi^n + (r-2)\Delta t \times \alpha, \tag{20}$$

$$\varphi^{n+r-1} = \varphi^n + (r-1)\Delta t \times \alpha, \tag{21}$$

where r, Δt, and α denote the number of future times, the time increment, and the piecewise approximate slope based on the result of the preliminary estimation, respectively.

Here, some problem should be noticed, such as how to discriminate the functional form of the undetermined source and why the type of piecewise constant slope was chosen in this case. It is not difficult to find the answer from the result of the preliminary estimation (see Fig. 2(a)). Moreover, the greater number of future times can be utilized to provide a considerably stable computational result and that is available to discriminate the functional form of the source strength. Additionally, if

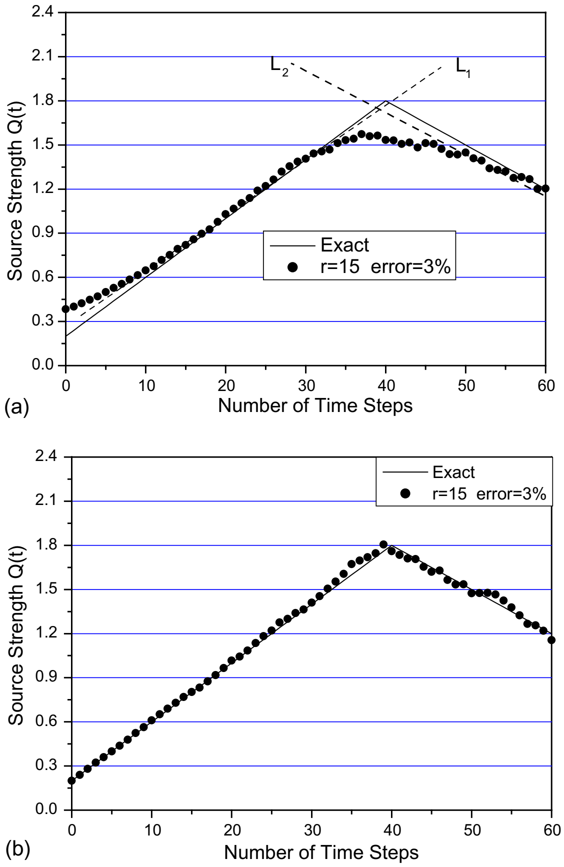


Fig. 2. The estimation in Case 3.1: (a) the preliminary estimation using SFSM, (b) the modified estimation using the modified sequential function specification method (MSFSM).

needed, computing the temperature distribution can recheck the modified result.

By substituting Eqs. (18)–(21) into Eqs. (7)–(10), Eqs. (12)–(15) are modified as follows:

$$T_i^n = u_i^0 + \gamma_i^n \varphi^n, \tag{22}$$

$$T_i^{n+1} = u_i^1 + \Delta t \times \alpha \times \gamma_i^n + \left(\gamma_i^{n+1} + \gamma_i^n\right)\varphi^n, \tag{23}$$

...

$$\begin{aligned} T_i^{n+r-2} = {} & u_i^{r-2} + (r-2)\Delta t \times \alpha \times \gamma_i^n + (r-3)\Delta t \times \alpha \times \gamma_i^{n+1} \\ & + \cdots + 2\Delta t \times \alpha \times \gamma_i^{n+r-3} + \Delta t \times \alpha \times \gamma_i^{n+r-2} \\ & + \left(\gamma_i^{n+r-2} + \gamma_i^{n+r-3} + \cdots + \gamma_i^{n+1} + \gamma_i^n\right)\varphi^n, \end{aligned} \tag{24}$$

$$\begin{aligned} T_i^{n+r-1} = {} & u_i^{r-2} + (r-1)\Delta t \times \alpha \times \gamma_i^n + (r-2)\Delta t \times \alpha \times \gamma_i^{n+1} \\ & + \cdots + 2\Delta t \times \alpha \times \gamma_i^{n+r-2} + \Delta t \times \alpha \times \gamma_i^{n+r-1} \\ & + \left(\gamma_i^{n+r-1} + \gamma_i^{n+r-2} + \cdots + \gamma_i^{n+1} + \gamma_i^n\right)\varphi^n. \end{aligned} \tag{25}$$

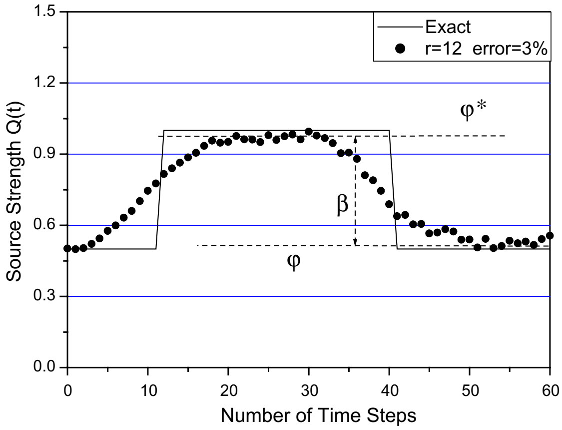


Fig. 3. The preliminary estimation by SFSM in Case 3.2.

Eqs. (22)–(25) are the modified sequential function specification algorithms. The closed approaching estimation of undetermined heat source strength can be obtained by using the linear least-squares error method in each time step, as shown in Fig. 2(b).

3.2. Case B: stepped source—slope → ∞ at some specific times

Fig. 3 shows the computational result of an undetermined stepped source by SFSM under conditions of 3% random measurement error and $r = 12$. Because of the constant heat source assumption, the leading error is appearing in the adjoining time steps in that the step response acting, and the number of affected time steps is with respect to the number of future time. The proposed modification is to find the approximate amplitude of step response based on the preliminary estimation over the time domain, except the time steps with significant

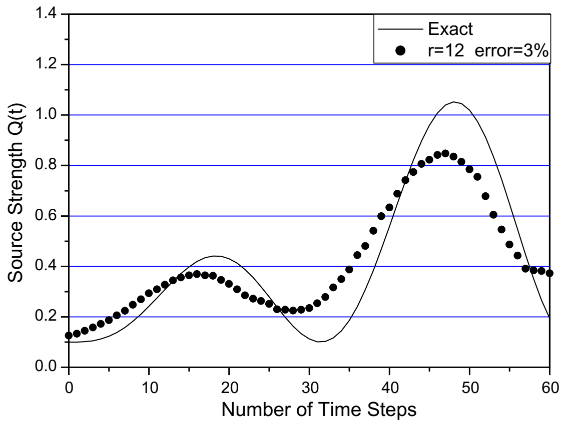


Fig. 4. The preliminary estimation by SFSM in Case 3.3.

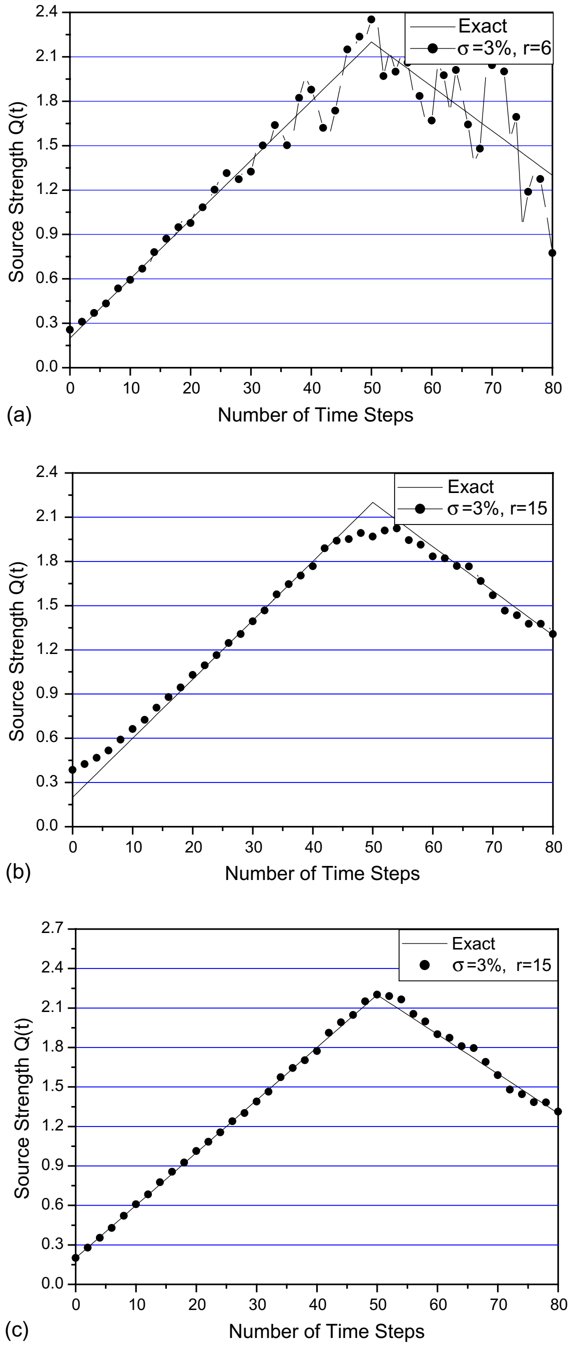


Fig. 5. The estimation in Example 1: (a) $r = 3$, insufficient future time steps; (b) $r = 15$, preliminary stable estimation by SFSM; (c) $r = 15$, modified estimation by MSFSM.

leading error. The correlation of estimated undetermined source can be modified as Eq. (26):

$$\varphi^* = \varphi + \beta, \tag{26}$$

where φ^* is the average strength over the time steps with the step response rising, φ is the average strength without the step response acting, and β is the average amplitude. Substituting Eq. (26) into Eqs. (7)–(10) and then using the linear least-squares error method is capable of obtaining a good modified result.

3.3. Case C: parabolic slope

Fig. 4 shows the computational result of an undetermined parabolic source by SFSM under conditions of 3% random measurement error and $r = 12$. In the case of source with a time-dependent slope (source strength with respect to time) is not appropriate to use the methods in Cases 3.1 and 3.2 to eliminate the leading error for two reasons. The first is that the procedure of curve-fitting solutions of the preliminary estimation would generally bring about a significant error. The other reason is that the exact slope of the function of the undetermined source at a specific time cannot be found from the information of the preliminary estimation because of the leading effect. The solution of preliminary estimation is available, however, and the correlation of the source strength at a specific time can be listed as Eq. (27). Consequently, one of the practicable approaches is to substitute Eq. (27) into Eqs. (7)–(10) and then utilizes the linear least-squares error method to obtain the modified result.

$$\varphi^{n+i} = \varphi^n + \Delta\varphi^{n+i} \quad (i = 1, 2, \ldots, r-1). \tag{27}$$

The other practicable approach is to use the concept of leading difference as Eq. (27) where m denotes the leading number of time steps and is less than $r - 1$. The optimal value can be obtained only several times of trial and error:

$$\Delta\varphi^{n+i} = \varphi^{n+i+m} - \varphi^{n+m} \quad (i = 1, 2, \ldots, r-1). \tag{28}$$

Table 1
The comparison of estimated strength of source in Example 1

Time step no.	Exact	SFSM		MSFSM
		$r = 6$	$r = 15$	$r = 15$
5	0.4	0.4099	0.4911	0.3913
10	0.6	0.5933	0.6622	0.6090
20	1	0.9725	1.0289	1.0139
30	1.4	1.4964	1.3933	1.389
40	1.8	1.6799	1.7668	1.7726
50	2.2	1.9617	1.9677	2.2007
60	1.9	1.5757	1.833	1.8996
70	1.6	1.110	1.5706	1.5893
80	1.3	1.4625	1.3066	1.3119
Average error		8.22%	7.05%	1.53%

$L1 = 0.2205 + 0.0190t$, $L2 = 1.8605 - 0.0153t$
($\alpha_1 = 0.0190$, $\alpha_2 = -0.0153$)

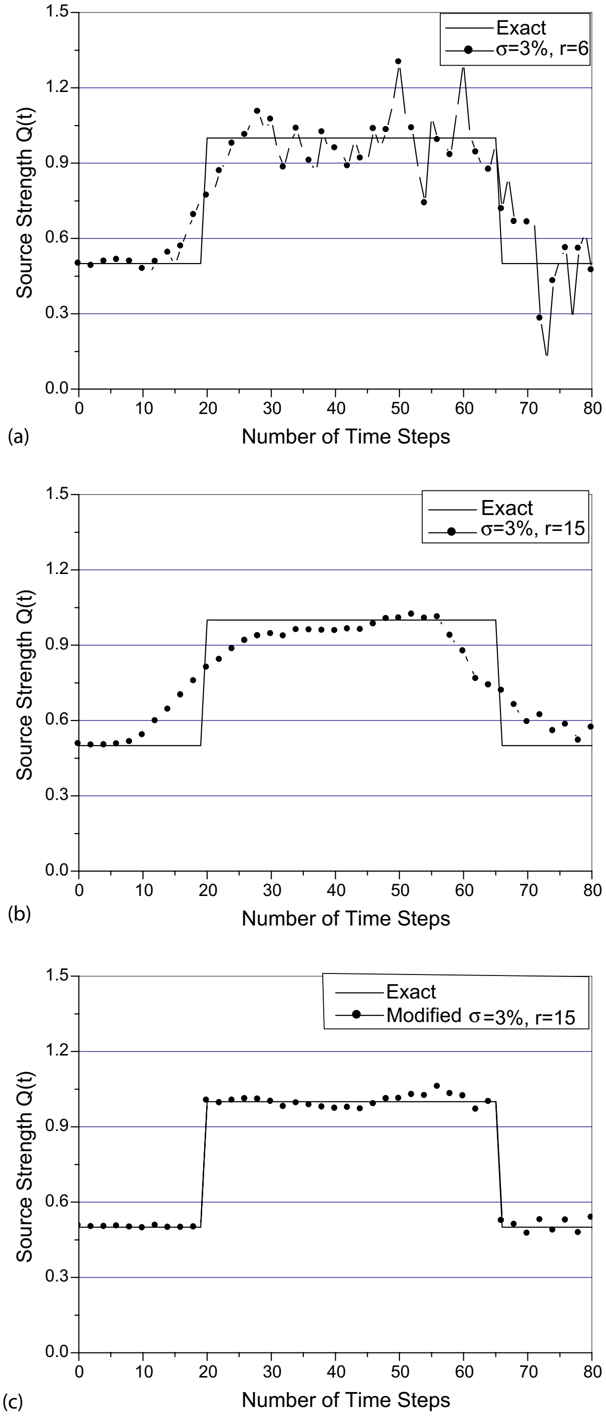


Fig. 6. The estimation in Example 2: (a) $r = 3$, insufficient future time steps; (b) $r = 15$, preliminary stable estimation by SFSM; (c) $r = 15$, modified estimation by MSFSM.

4. Results and discussion

In this section, four cases are described in which have various functional forms of the source are used to demonstrate that the proposed method can be employed to solve the IHCP and yield stable and accurate computational results. All the examples assume that one undetermined time-dependent source acts at the center of the observed object. For numerical computation, the dimensionless length of this object is unity. Meanwhile, the computational spatial increment is 0.1 and the time increment is 0.02. The measured position is set on one of the boundaries of the object. Additionally, the simulated temperature in this study is generated from the exact temperature and is assumed with random measurement errors, as shown in Eq. (29).

$$Y_i^n = T_i^n(1 + \lambda \times \sigma), \tag{29}$$

where T_i^n and Y_i^n denote the exact and measured temperature at spatial gird $x = x_i$ and temporal gird $t = t^n$, respectively. The term σ is the standard deviation of the measurement errors and λ is a random real number between −1 and 1, so that Eq. (29) describes an unpredictable error in the measurement of temperature.

Example 1. This example is to estimate the strength of one triangular source. Here, 3% random measurement error is considered. The functional form of heat source is presented as follows:

$$Q(t) = 0.2 + 2t, \quad 0 \leqslant t \leqslant 1.0,$$

$$Q(t) = 2.2 - 1.5t, \quad 1.0 < t.$$

About the form and strength of the undetermined source, we know nothing before the inverse algorithm is performed. Consequently, the preliminary estimation is operated firstly, under the conditions of the temporary assumption of a constant strength of heat source and $r = 6$, and the sequential algorithms that were derived in the above section are applied. Fig. 5(a) shows the computational result is fluctuant and so a greater number of future time steps should be employed. Obviously, the result displayed in Fig. 5(b), in which $r = 15$ is used, is much more stable than that in Fig. 5(a), but the leading error is also enlarged comparatively. To improve the accuracy and reliability, the leading error must be eliminated under a stable solution.

The proposed modified method is applied to improve the inconsistent characteristic. From Fig. 5(b), it could be convinced that the undetermined source might be a piecewise linear function. Hence, based on the result of the preliminary estimation, apply the modified algorithm in Case 3.1 to obtain two piecewise approximate slopes $\alpha_1 = 0.0190$ and $\alpha_2 = 0.0153$ with $r = 15$. These approximate slopes are available and are substituted into Eqs. (22)–(25). A accurate estimation can then be obtained readily. This result is shown in Fig. 5(c).

Table 1 displays a comparison of the computational results for Example 1 that were obtained using various conditions. It is clear that the average error by SFSM with $r = 6$ and $r = 15$ are 8.22% and 7.05%, respectively. When $r = 15$, the computational result is more stable than that of $r = 6$, but the average error is not better; on the contrary, it is worse, as a result of the leading error. The above two cases verify the inconsistent phenomenon mentioned earlier. Comparatively, the proposed method provides a highly accurate and stable estimation with an average error of only 1.53%.

Example 2. This example describes the estimation of the undetermined strength of source with a stepped characteristic. Similarly, 3% random measurement error is also considered here, as it was in Example 1. The heat source is assumed as follows:

$$Q(t) = 0.5 \quad 0 \leqslant t$$

$$Q(t) = 1.0 \quad 4 \leqslant t \lesssim 1.3$$

$$Q(t) = 0.5 \quad 1.3 \leqslant t$$

Fig. 6(a) displays the fluctuant computational result obtained when the insufficient future times ($r = 6$) are used. Fig. 6(b) presents the stable computational result obtained when a greater number of future time ($r = 15$) is used, but the leading error also appears to be significant in the adjoining time steps of 20th time step and 65th time step. Such error must be eliminated reasonably. From the result of preliminary estimation as shown in Fig. 6(b), an unit source form is supposed and the amplitude of step response of source can be evaluated and it equals 0.4712. This result is available to modify the leading error as in the description of Case 3.2. The modified result is presented in Fig. 6(c) and it is much better than the result in Fig. 6(b) as the leading error is fully eliminated. Table 2 displays a comparison between the computational results for Example 2 that are obtained by the SFSM and MSFSM. The average

Table 2
The comparison of estimated strength of source in Example 2

Time step no.	Exact	SFSM		MSFSM
		$r = 6$	$r = 15$	$r = 15$
10	0.5	0.5052	0.5398	0.5027
15	0.5	0.4919	0.6789	0.4997
20	1	0.7694	0.8012	1.0036
30	1	1.0726	0.9472	1.0024
40	1	0.9581	0.9656	0.9780
50	1	1.1368	1.0186	1.0078
60	1	1.3022	0.8915	1.0126
70	0.5	0.6640	0.5974	0.4855
80	0.5	0.4729	0.5802	0.5301
Average error		15.87%	15.12%	2.43%

$\varphi^* = 0.9854, \quad \varphi = 0.5142, \quad \varphi^* - \varphi = 0.4712$

error is reduced significantly to only 2.43% when using the MSFSM.

Example 3. This example discusses an estimation of the undetermined strength of heat source with a time-

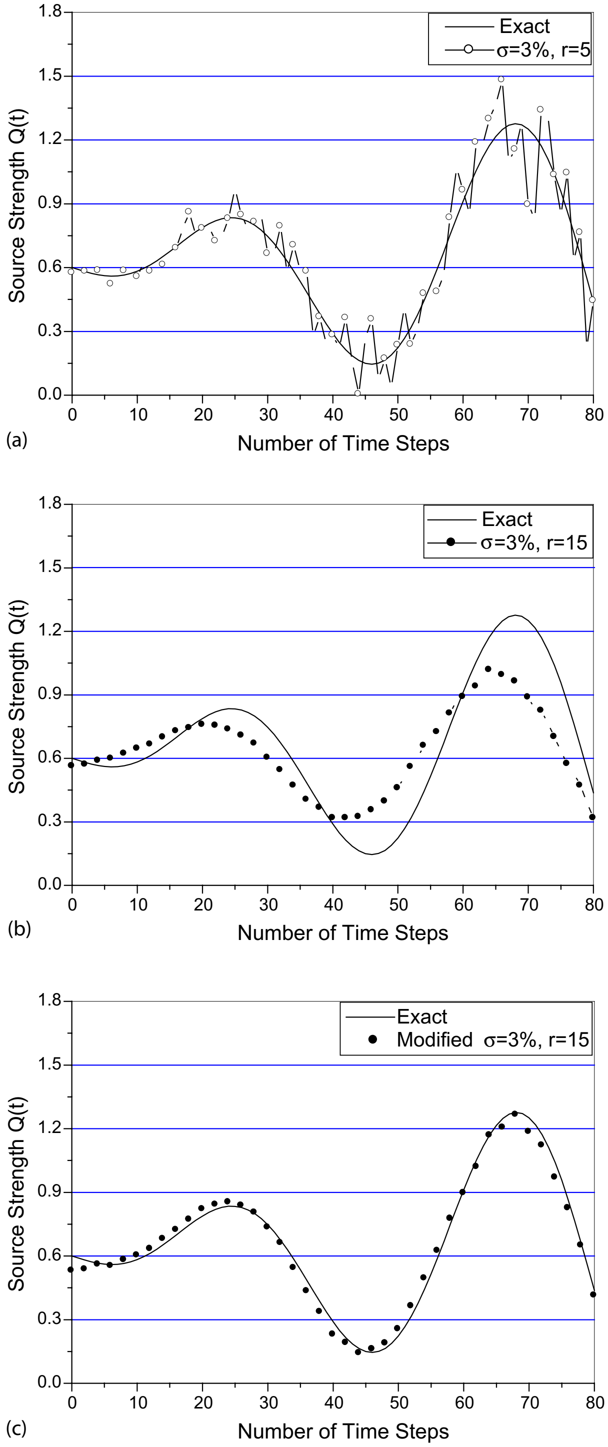


Fig. 7. The estimation in Example 3: (a) $r = 3$, insufficient future time steps; (b) $r = 15$, preliminary stable estimation by SFSM; (c) $r = 15$, modified estimation by MSFSM.

Table 3
The comparison of estimated strength of source in Example 3

Time step no.	Exact	SFSM		MSFSM
		$r = 5$	$r = 15$	$r = 15$
5	0.56176	0.5429	0.5950	0.5488
10	0.583	0.5566	0.6454	0.6022
20	0.78844	0.7837	0.7583	0.8198
30	0.74708	0.6652	0.6032	0.7337
40	0.28977	0.2815	0.3172	0.2391
50	0.22305	0.2347	0.4589	0.2553
60	0.91157	0.9628	0.8898	0.8959
70	1.2513	0.8957	0.8873	1.1838
80	0.4376	0.4437	0.3172	0.4132
Average error		16.99%	27.98%	4.21%

Table 4
The comparison of estimated strength of source in Example 4

Time step no.	Exact	SFSM	MSFSM
		$r = 24$	$r = 24$
5	0.4	0.6243	0.4014
10	0.6	0.7434	0.5873
20	1	1.1429	1.0665
30	1.4	1.4179	1.3817
40	1.8	1.7535	1.8221
50	2.2	1.7941	2.1814
60	1.9	1.6893	1.8792
70	1.6	1.4120	1.5050
80	1.3	1.2643	1.3099
Average error		17.82%	2.65%

$L1 = 0.3451 + 0.0193t$, $L2 = 1.9204 - 0.0147t$
($\alpha_1 = 0.0193$, $\alpha_2 = -0.0147$)

dependent slope (source strength vs. time) characteristic. Similarly, 3% random measurement error is also considered in this case. The heat source is assumed as follows:

$$Q(t) = 0.6 + 0.5t\sin\left(7t - \frac{\pi}{2}\right), \quad 0 \leqslant t.$$

Fig. 7(a) displays the fluctuant computational result that is caused by an insufficient future times used ($r = 5$). A much more stable computational result is obtained when a greater number of future times ($r = 15$) is used, but the leading error is also enlarged comparatively. From Fig. 7(b), the undetermined source conceivably could be that a curve function, such as, a polynomial, exponential, or trigonometric function. Consequently, the modified procedures in Case 3.3 are applied and the result is displayed in Fig. 7(c). Obviously, it is considerably more accurate and stable than the result displayed in Fig. 7(b). Moreover, the computational process is also efficient. Table 3 displays a comparison between the computational results. The average error is clearly reduced from 16.99% or 27.98% to 4.21% by the proposed method.

Example 4. In this example, the functional form of the heat source strength is the same as that of Example 1, but the random measurement error is 10%. It is noteworthy that most of the presented studies in solving the unknown parameters for IHCP are chosen with slight random measurement error as a result of preventing to obtain an unstable solution resulted from ill-posed problem. Nevertheless, some factors in practical application might result in significant measurement errors because of, for example, obstructed or restrained measurement, and the sensitivity of instrument. In this example, $r = 24$ is utilized to stabilize the computational result by SFSM, and results in significant leading error. Fig. 8 displays that a highly accurate and stable estimation can still be obtained by the proposed method,

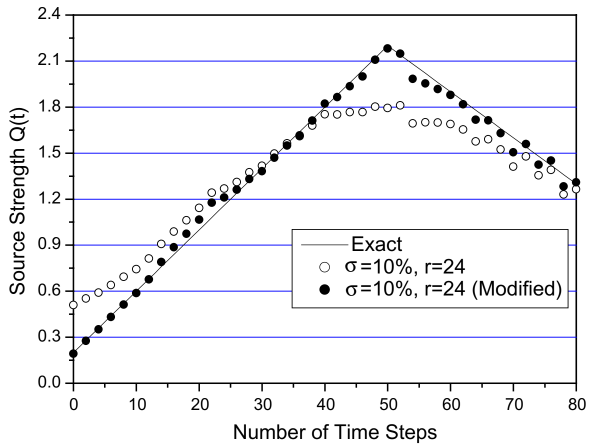


Fig. 8. The comparison of estimation result for SFSM and MSFSM in Example 4, 10% random measurement error considered.

even though the measurement error is not slight. This improvement is also displayed clearly in Table 4. The average error is reduced from 17.82% to 2.65%.

5. Conclusion

The SFSM is an effective method for solving the IHCP. When the measurement error is considered, this method is needed to combine with several future times to stabilize the solutions. This process, however, causes leading errors to appear and therefore decrease the accuracy of the solutions. Additionally, the leading error becomes progressively larger upon increasing the number of future times. This phenomenon restricts the performance and efficiency of the SFSM.

This paper proposes a modified sequential method that can deal with the determination of unknown parameter for the IHCP efficiently and obtain a highly accurate result. There are two essential procedures in the proposed method: A preliminary estimation is first performed by the use of the inverse sequential algorithms as Eqs. (12)–(15) and the linear least-squares error method as Eq. (17). The next, based on the result of preliminary estimation applies the modified procedures that proposed in this study to obtain an anticipated estimation.

Four examples, with various functional forms of the undetermined heat source strength used in the first three and larger measurement error used in the fourth, to demonstrate the effectiveness, practicality, and reliability of the proposed method. These functional forms include a piecewise linear, a unit step, and a trigonometric function. The results verify that the proposed method is capable of solving the undetermined parameter for various functional or mixed forms of the IHCP.

The main contribution of this study is a demonstration that the proposed method can be applied to solve the IHCP under the inherent advantage of good computational efficiency for the sequential method, and provide a highly accurate and stable solution. Even the random measurement error is not slight, for instance, 10% or more, the proposed method is still able to provide a considerably acceptable estimation.

The proposed method can be applied widely to other inverse problems and provide a more accurate estimation relative to those of previous methods. Certainly, two- and three-dimensional inverse problems can also be applied in similar manners.

References

[1] J.V. Beck, B. Blackwell, C.R. St. Clair, Inverse Heat Conduction: Ill-Posed Problem, Wiley, New York, 1985.

[2] O.M. Alifanov, Inverse Heat Transfer Problem, Springer-Verlag, Berlin, 1994.

[3] A.N. Tikhonov, V.Y. Arsenin, Solutions of Ill-Posed Problems, Winston, Washington, DC, 1977.

[4] J.V. Beck, Nonlinear estimation applied to the nonlinear inverse heat conduction problem, Int. J. Heat Mass Transfer 13 (1970) 703–716.

[5] C.Y. Yang, A sequential method to estimate the strength of the heat source based on symbolic computation, Int. J. Heat Mass Transfer 41 (14) (1998) 2245–2252.

[6] C.Y. Yang, The determination of two heat sources in an inverse heat conduction problem, Int. J. Heat Mass Transfer 42 (1999) 345–356.

[7] C.Y. Yang, Inverse estimation of mix-typed boundary conditions in heat conduction problems, J. Thermophys. Heat Transfer 12 (4) (1998) 552–561.

[8] E. Videcoq, D. Petit, Model reduction for the resolution of multidimensional inverse heat conduction problems, Int. J. Heat Mass Transfer 44 (2001) 1899–1911.

[9] T.-G. Kim, Z.-H. Lee, Time-varying heat transfer coefficients between tube-shaped casting and metal mold, Int. J. Heat Mass Transfer 40 (15) (1997) 3513–3525.

[10] S. Chantasiriwan, Inverse heat conduction problem of determining time-dependent heat transfer coefficient, Int. J. Heat Mass Transfer 42 (2000) 4275–4285.

[11] J.V. Beck, B. Blackwell, A. Haji-sheikh, Comparison of some inverse heat conduction methods using experimental data, Int. J. Heat Mass Transfer 39 (17) (1996) 3649–3657.

[12] E. Videcoq, D. Petit, A. Piteau, Experimental modeling and estimation of time varying thermal sources, Int. J. Thermal Sci. 42 (2003) 255–265.

[13] H.-J. Reinhardt, Analysis of sequential methods of solving the inverse heat conduction problem, Numer. Heat Transfer, Part B 24 (1993) 455–474.